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The Great Cluster in Hercules. By W. E. Plummer, M.A.

Some time since I received from Professor G. Hale, the late Director of the Yerkes Observatory, a photograph of the cluster in *Hercules*, taken with the large refractor. This plate is labelled: "Messier 13, photographed with the 40-inch Yerkes refractor, August 15, 1900. Exposure four hours, $8^h\ 30^m$–$12^h\ 30^m$ Central Standard Time, with double-slide plate-holder, yellow screen, isochromatic plate. (Signed) G. W. Ritchey." Having recently measured this plate, it seemed desirable to compare the results with what earlier measures existed, and to trace, if possible, any change in the relative coordinates of the stars forming the group. Such an inquiry is no doubt premature, but it might serve to show whether ten or a hundred years were needed in order to conduct such investigations with success.

At the time of the arrival of the plate I had no suitable measuring machine, but Professor Turner kindly placed at my disposal one that had been used at Oxford in the measurement of the plates of the International Chart. The American plate is of the ordinary quarter-plate size and without any réseau lines; while the Oxford machine, without a special adaptor, is only available for plates of the size of those recommended for the International Chart, and the method of measurement requires the employment of a réseau. It was therefore necessary to transfer the American negative to a plate of convenient size and marked with the réseau. For this service I was again indebted to the Oxford Observatory. The copy appears to have been very successfully effected. It might have been anticipated that in the two transfers necessary to produce a fresh negative many images of the fainter stars would be lost; but this is not so. In the case of the faintest stars visible on the original negative, I could generally trace them on the copy by the aid of allineation, and in only a few cases was the image too faint for measurement.

This transferred negative has been measured in two reversed positions of the plate, and with the scales and method of measurement employed the greatest accuracy attainable is the thousandth part of the distance between two réseau lines. Owing to the great focal length of the Yerkes refractor, the angle subtended between two of these lines is approximately 53″, so that practically the measures have been made to $\frac{1}{20}''$. The diameters of the images were also measured, to determine the magnitude of the stars; but, as will be seen in the sequel, this portion of the work is not so satisfactory as could have been wished. The total number of objects measured is 2131, distributed over an area of eleven minutes square. The law of distribution throughout this area will be considered later.

To reduce the measurements to standard coordinates a certain number of fiducial stars is necessary, and within the small area

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of the American negative, meridian places of these do not exist. It was therefore necessary to connect the Yerkes plate with one embracing a larger area. This plate was taken by Mr. H. C. Plummer and is discussed in the *Monthly Notices* of 1904, November. The reductions of the measures have been based upon the positions of the stars there given. Of the seventy stars whose places are recorded some sixty-two are common to both plates; but in a few instances the stars selected at Oxford for measurement were found to have a companion so close that the two images would, from the shorter focal length, naturally coalesce, and the centre of the combined image would not correspond to the centre of either, or necessarily to the mean of the two. On this ground seven stars were excluded from the comparison; and this process of rejection, for the reasons mentioned, might have been carried further with advantage. An inquiry, conducted in the usual way, showed that the Liverpool (L) measures could be connected with the Oxford (O) by means of the formulæ

$$x_O = +0{\cdot}89584x_L + 0{\cdot}00061y_L - 8{\cdot}0234$$
$$y_O = -0{\cdot}00033x_L + 0{\cdot}89556y_L - 14{\cdot}9226$$

Besides these Oxford measures, differential coordinates of a number of stars have also been given by Professor Scheiner, in his paper entitled "Der grosse Sternhaufen im *Hercules* Messier 13 nach Aufnahmen am Potsdamer photographischen Refractor" (*Abhandlungen Königl. Preuss. Akad. der Wissenschaften zu Berlin*, 1892). The places given in this paper are referred to a star, approximately in the centre of the group, whose coordinates for 1891·0 are quoted as

$$\alpha = 16^{h}\ 37^{m}\ 46^{s}{\cdot}85 \qquad \delta = +36^{h}\ 40^{m}\ 22^{s}{\cdot}9$$

This position, reduced to 1900·0, differs from the assumed centre of the Oxford plate by (P—O)

$$x = +1'\ 13''{\cdot}58$$
$$y = -\ \ 40{\cdot}40$$

For the purposes of comparison these constants have been applied to the Potsdam measures, together with the small differential corrections due to precession, in order to refer the measures to the equinox 1900·0. This was certainly not the best method of comparing the different sets of measures; but as it was the course that was actually pursued the results are set down here, as they show both the character of the agreement that may be expected between the measures and the necessity of adopting a more legitimate and accurate method of comparison:—

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Table I.

Comparison of Measures of Potsdam, Oxford, and Liverpool.

Potsdam No.	Liverpool *x* 1900.	L−P.	Oxford No.	L−O.	Liverpool *y* 1900.	L−P.	L−O
	′ ″	″		″	′ ″	″	″
7	−3 55·27	−0·26	7	−0·07	+4 52·68	−0·37	0·00
9	−3 52·57	+0·25	8	−0·13	−5 13·56	−0·58	−0·54
10	−3 48·71	−0·09	9	−0·05	+ 55·05	+0·04	+0·63
14	−3 29·89	+0·14	11	−1·45	−2 41·83	−0·70	−0·19
27	−2 57·83	+1·54	13	+0·61	−1 53·97	−0·13	+0·45
29	−2 56·45	+0·17	14	+0·19	−3 52·07	−0·78	−0·17
30	−2 54·16	+0·24	15	+1·40	+2 35·55	−0·80	+0·09
34	−2 39·68	−0·02	16	−0·5	− 31·65	−0·04	+0·39
47	−2 7·84	+0·03	17	−0·82	+1 46·49	−0·56	+0·89
48	−2 4·29	+0·44	18	+0·09	−3 0·13	−0·45	−0·13
51	−1 57·43	+0·51	19	−0·01	−2 45·87	−0·46	−0·69
56	−1 38·20	+0·63	21	+0·92	−5 41·34	−0·80	−0·66
63	−1 26·50	+0·06	22	−0·34	− 31·82	−1·58	−0·62
68	−1 19·91	−0·19	23	+0·85	+5 25·84	−0·42	+0·52
72	−1 15·25	+0·28	24	+0·71	−3 3·81	−0·47	−0·21
92	− 51·78	+0·05	25	+0·18	+3 24·73	−0·12	+0·73
103	− 43·70	+0·08	26	+0·22	+1 11·58	−0·90	+0·12
109	− 39·85	+0·20	27	−0·01	−2 33·00	−0·43	−0·18
127	− 26·87	+0·10	28	−0·35	− 1·90	−0·53	+0·20
129	− 23·54	+0·19	29	+0·34	− 57·21	−0·80	−0·21
130	− 23·04	+0·39	30	−0·48	−3 49·30	−0·16	+0·32
148	− 8·16	+0·47	31	−0·54	−2 24·22	−0·30	+0·44
216	+ 19·76	+0·14	32	−0·16	− 43·74	−0·44	−0·12
227	+ 22·55	+0·32	33	−0·07	−3 21·97	−0·18	+0·35
234	+ 28·27	+0·13	34	−0·05	+2 19·72	+0·03	+0·76
287	+ 49·05	+0·08	35	+0·27	− 2·71	−0·63	−0·49
288	+ 50·10	+0·55	36	+0·06	−5 52·50	−0·05	−0·24
296	+ 52·11	+0·29	37	−0·09	−2 54·67	−0·31	−0·07
418	+1 23·44	+0·74	38	−0·08	+ 16·72	−0·68	+0·88
479	+1 40·43	+0·70	40	+0·59	−2 4·87	−0·12	+0·05
507	+1 49·42	+0·81	41	+0·76	−2 6·97	−0·96	−0·73
529	+1 53·87	+0·35	42	−0·37	−5 59·43	+0·02	+0·39
594	+2 15·02	+0·01	43	−0·94	+2 33·56	−0·07	+0·02
603	+2 16·48	+0·03	44	−0·20	+1 37·46	−0·51	−0·28
602	+2 16·58	+0·15	45	−0·46	+1 24·89	−0·52	−0·07
642	+2 32·71	+0·03	46	−0·29	−2 36·24	−0·09	+0·18

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Potsdam No.	Liverpool x 1900.	L−P.	Oxford No.	L−O.	Liverpool y 1900.	L−P.	L−O.
	′ ″	″		″	′ ″	″	″
654	+2 39·46	+0·23	47	+0·22	− 5·59	−0·55	−0·19
726	+3 11·37	+0·45	48	+0·21	−4 18·48	+0·08	−0·12
727	+3 11·88	+0·44	49	+0·12	−4 52·52	−0·15	+0·34
731	+3 14·03	+0·34	50	+0·35	+1 8·72	−0·70	+0·32
739	+3 21·11	+0·24	51	−0·01	+4 43·74	−0·90	−0·60
749	+3 34·33	+0·20	53	+0·01	−2 5·58	−0·19	+0·18
748	+3 34·15	+0·81	54	−0·41	−5 54·40	−0·45	+0·26
771	+4 5·18	+0·50	55	−0·34	−1 12·02	−0·54	+0·34
774	+4 14·24	+1·37	56	+0·56	+ 26·78	+1·37	+0·50
778	+4 18·76	+0·06	57	−0·14	+4 51·56	−0·51	−0·28
781	+4 30·94	+0·52	58	+0·70	+1 27·40	−0·56	−0·26
785	+4 36·99	+0·33	59	−0·21	−3 13·61	−0·63	+0·07
786	+4 37·78	−0·04	60	−0·38	+5 19·78	−0·29	−0·14
795	+5 2·93	+0·20	61	+0·23	− 49·26	−0·44	−0·12
799	+5 18·38	+0·58	62	−0·10	−5 1·65	−0·11	−0·45
805	+5 37·44	+0·25	63	0·00	+3 13·11	−0·63	−0·57
815	+6 44·10	+0·14	64	+0·78	+1 0·21	−0·49	+0·09
821	+7 6·50	+1·05	65	−0·10	−6 10·61	−0·37	−0·29

Note.—Star No. 30 Potsdam has another star with which it might coalesce. No. 60 is ill-formed on the Yerkes plate, and has not been used in deriving the plate constants. No. 418 is described by Scheiner as "neblig." It has a small star quite close. No. 748 has the same note, and is similarly circumstanced. No. 774 is marked "Duplex oder nebelknoten." This is an instructive case. There is another star distant 1″·36 in x and 2″·69 in y. Neither star agrees very well with the Potsdam place, but the mean of the two would be satisfactory.

The few changes of sign in the residuals (L−P) disclose a systematic difference between the two sets of measures, while the uniformity in the amount offers little hope of tracing any effects of motion among the stars. The larger part of the discrepancies may be removed by a slight alteration in the assumed mean distance between the origin of coordinates on the Liverpool (or Oxford) and Potsdam plates. The mean difference between Liverpool and Potsdam, as determined from these fifty-four stars, is (L—P)

	″
x	+0·32
y	—0·41

It is not difficult to suggest various ways in which this difference might arise; but that it is a real difference and does not arise from the use of the outlying and brighter stars of the group is shown by a more extended comparison between the stars that have been measured at Potsdam and Liverpool. I proposed

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to compare the whole of the Potsdam measures with those derived from the Yerkes plate, but in the denser part of the cluster it has not always been possible to identify with certainty the star that has been measured by Professor Scheiner; and, in addition to these instances of defective identification, some other stars have been rejected because they are accompanied by a bad note implying uncertainty in the original measures. The total number of stars between which it seemed safe to institute comparisons is 560, and the general result is shown in Table II., where a zone is to be understood as an arc of declination between two consecutive réseau lines on the copy of the Yerkes plate.

TABLE II.

(L−P).

Zone.	Approx. Distance of Centre of Zone from Centre of Group.	Number of Stars Compared.	Mean Difference of x.	Mean Difference of y.
	′ ″		″	″
1	−5 44	12	+0·44	−0·20
2	−4 50	14	+0·57	−0·17
3	−3 57	28	+0·43	−0·22
4	−3 3	62	+0·32	−0·33
5	−2 9	86	+0·29	−0·37
6	−1 15	103	+0·32	−0·39
7	−0 22	95	+0·16	−0·52
8	+0 31	76	+0·19	−0·62
9	+1 25	37	+0·09	−0·64
10	+2 19	23	+0·14	−0·63
11	+3 13	12	+0·16	−0·34
12	+4 6	6	+0·11	−0·54
13	+4 59	4	+0·30	−0·35

Here again the mean difference is +0″·27 and −0″·41 in x and y respectively, practically the same as before, with some indications of regular systematic progress. But the systems by which the measures have been reduced in the two cases are essentially different; and before deciding to what extent the discrepancy in the measures is real it will be necessary to reduce the measures to some uniform plan. There are two methods which seem available: either to introduce a scale value into Dr. Scheiner's work which shall be the same as that employed in the Oxford measures, and to re-reduce the position of his normal star, or to derive the constants of the Potsdam plate *de novo*, and in the same manner and from the same stars that were used in deriving the constants of the Yerkes plate.

With regard to the first proposal it is to be remarked that Dr. Scheiner had at his command the places of seven stars out of the Lund zones, and these places, or some of them, have been used for the determination of the scale. From stars A.G.C. Lund 6829 and 6863 he has derived one value of the scale or réseau interval, and from Lund 6836 and 6868 he has derived

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another; and since he has two plates taken on consecutive evenings he obtains four results :—

	From the First Plate.	From the Second Plate.
1st pair of stars	$299''{\cdot}87$	$299''{\cdot}86$
2nd pair of stars	$299{\cdot}96$	$299{\cdot}97$
Mean	$299{\cdot}92$	$299{\cdot}92$

This mean result looks very satisfactory; but when we introduce the corrections (*Monthly Notices*, 1904 November, vol. lxv. p. 81) to the Lund stars, we get quite as accordant results though the means differ noticeably :—

	From the First Plate.	From the Second Plate.
1st pair of stars	$299''{\cdot}843$	$299''{\cdot}821$
2nd pair of stars	$299{\cdot}868$	$299{\cdot}888$
Mean	$299{\cdot}855$	$299{\cdot}855$

The position of the star of reference has been derived from A.G.C. Lund 6829 and 6868, and from A.G.C. Lund 6836 and 6857. The measured angles Δ, given on page 19, I have reduced in the ratio of 299·855 : 299·92; but I have used unchanged the angle p (the angle between the lines drawn from the star of reference to one pair of the Lund stars), and have again brought in the corrections to the places of the Lund stars given in the *Monthly Notices* already referred to. The positions of the star of reference as given by Dr. Scheiner are

	Plate I.		Plate II.	
1st pair	$\alpha = 249^{\circ}\ 27'\ 39''{\cdot}3$	$\delta = 36^{\circ}\ 41'\ 5''{\cdot}0$	$\alpha = 249^{\circ}\ 27'\ 35''{\cdot}1$	$\delta = 36^{\circ}\ 41'\ 1''{\cdot}2$
2nd pair	$\alpha = 249\ 27\ 39{\cdot}3$	$\delta = 36\ 41\ 4{\cdot}7$	$\alpha = 249\ 27\ 34{\cdot}3$	$\delta = 36\ 41\ 0{\cdot}8$

The new values for the same star are

1st pair	$\alpha = 249^{\circ}\ 27'\ 39''{\cdot}21$	$\delta = 36^{\circ}\ 41'\ 4''{\cdot}71$	$\alpha = 249^{\circ}\ 27'\ 34''{\cdot}99$	$\delta = 36^{\circ}\ 41'\ 0''{\cdot}52$
2nd pair	$\alpha = 249\ 27\ 39{\cdot}69$	$\delta = 36\ 41\ 4{\cdot}90$	$\alpha = 249\ 27\ 34{\cdot}75$	$\delta = 36\ 41\ 0{\cdot}69$

The mean difference in α is $+0''{\cdot}27$ (or in $x + 0''{\cdot}22$) and in $\delta - 0''{\cdot}41$; a very close agreement with the quantity shown to be necessary to remove the constant error between the two sets of measures.

The same result is practically arrived at if the Potsdam measures are compared, not directly with the Lund stars, but with the places derived in the *Monthly Notices*. Taking the same list of stars as that previously employed, page 803, the difference between Oxford and Potsdam is given by the expression

$$x_O - x_P = +0''{\cdot}029 x_P - 0''{\cdot}107 y_P + 1'\ 13''{\cdot}87$$
$$y_O - y_P = +0''{\cdot}092 x_P - 0''{\cdot}042 y_P - \ 40{\cdot}83$$

These formulæ include the effects of (1) difference of adopted plate centres, (2) difference of central meridian, (3) difference of mean equinox, (4) changes in (*a*) scale value and (*b*) orientation

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arising from the different systems of reduction. It is unnecessary to seek in detail to what extent the total coefficients are accounted for by these sources of difference, so far as they can be separately estimated. But the coefficients +0″·029 and −0″·042, which represent the total difference of scale value (per minute of arc) in the direction of the two coordinates, call for remark. No geometrical reason can be suggested for the difference, and the magnitude of the discrepancy (0″·071) is too great to be attributed with any assurance to the influence of accidental error. Hence, although the proof is by no means convincing, there is a slight suggestion that the form of the cluster as a whole is changing either by an expansion in the direction of x or a contraction in the direction of y.

Introducing these corrections into the Potsdam Catalogue, the residuals in Table II. have the following values :

Zone.	Mean Diff. of x.	Mean Diff. of y.	Zone.	Mean Diff. of x.	Mean Diff of y.
	″	″		″	″
1	−0·06	+0·02	8	−0·05	−0·14
2	−0·14	+0·07	9	−0·06	−0·07
3	−0·07	+0·04	10	+0·03	−0·06
4	+0·02	−0·11	11	0·00	+0·25
5	−0·04	+0·19	12	+0·01	+0·04
6	+0·01	−0·02	13	+0·06	+0·27
7	+0·03	+0·05	...	...	...

The smallness of the residuals shows, as was expected, that the interval of time is far too short to detect any relative motion in the stars composing the system. But the method of grouping the stars is not without its objections. It is probable that in a spherical cluster, seen projected on a plane, many of the stars apparently near each other are on opposite sides of the system and moving in opposite directions ; consequently in such cases the motions would have a tendency to counteract each other.

Magnitude.

The determination of magnitude is one of the least satisfactory parts of the inquiry. Not only is there wanting sufficient data to assist in the conversion of the measured diameters into conventional magnitudes, but the measurement of the diameters themselves is not very satisfactory. So long as the images present a well-blackened area the measurement is fairly easy, but in the faintest stars the blackening is not continuous. They consist of a certain number of darkened points which do not coalesce, and there is probably a tendency to make the measurements too large. Or if the exposure had been continued, producing still fainter stars on the plate, the images that have been measured would not have increased in diameter, but the internal parts would have become blacker. Practically, therefore, it is to

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be feared that there are two scales, one referring to the brighter, the other to the fainter stars.

Inasmuch as the total number of stars measured is roughly 2½ times that of Scheiner's Catalogue, it might be assumed that the Yerkes plate registers the positions of stars one magnitude fainter than those on the Potsdam plates. This would imply the fifteenth magnitude; but it is hardly probable that a law which has been found to hold roughly among the brighter and scattered stars in the heavens would be maintained here in orders of magnitude which have not hitherto been considered. The increase of number with decrease of magnitude must break down some time, and apart from this, there is some evidence that the distribution of stars of various magnitudes is not uniform. In Scheiner's catalogue the stars which are described as faintest are not those which occur most frequently. He has adopted practically seven classes of magnitudes, and in the case of those stars which it is possible to compare with the Liverpool measures the percentages in each class are as follows:

Magnitude.	Percentage.	Magnitude.	Percentage.
12·4	5	13·0	26
12·5	7	13·5	20
12·7	10	14	17
12·8	15		

This peculiarity in the falling off in numbers is not shown on the Yerkes plate, but another is manifest for which there appears to be no explanation in the way of observation, and which points with equal force to a want of uniformity in the distribution of the stars in each magnitude. Arranging the stars in order of diameter in successive groups, varying by five units (which corresponds approximately to 0″·3) we get the following numbers in each group:

TABLE IV.

Number of Stars in each Zone, arranged according to Diameter.

Distance from Centre.	Under 15.	15–20.	20–25.	25–30.	30–35.	35–40.	40–45.	45–50.	50–55.	55–60.	Over 60.	Objects not measurable.	Total.
−5·44	7	5	2	1	...	...	1	...	...	...	1	...	17
−4·50	18	4	...	3	1	2	1	3	...	1	...	...	33
−3·57	12	9	3	3	3	1	1	...	...	...	2	4	38
−3·3	24	11	6	5	2	3	2	1	...	1	5	11	71
−2·9	45	32	9	4	7	7	4	5	3	1	6	15	138
−1·15	144	64	33	10	11	13	12	8	3	5	13	22	338
−0·22	217	108	67	41	21	42	22	21	6	21	29	38	633
+0·31	214	92	45	30	10	26	13	7	6	3	16	10	472
+1·25	92	39	22	7	3	9	13	3	1	3	2	1	195
+2·19	45	15	10	3	...	4	4	1	2	2	4	...	90
+3·13	17	19	4	1	...	3	1	...	1	...	1	...	47
+4 6	11	11	1	5	1	1	...	1	...	...	1	...	32
+4·59	10	1	...	3	1	...	...	...	...	...	...	...	15
+5·52	5	2	2	1	...	...	...	1	1	...	...	...	12
Total	861	411	204	117	60	111	75	51	23	37	80	101	2131

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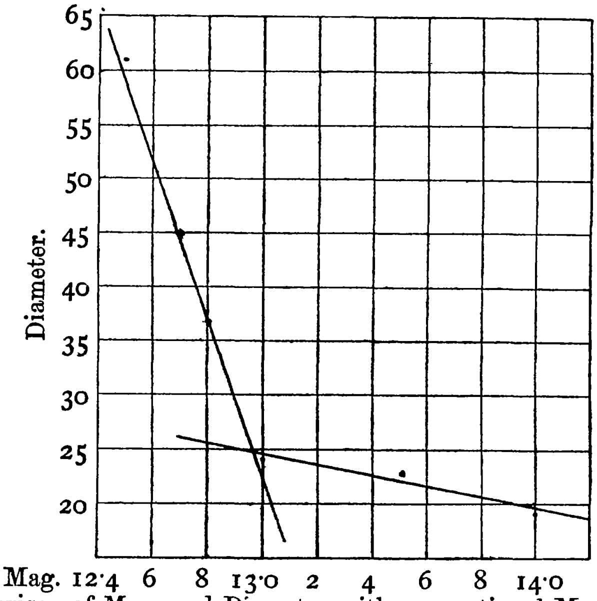
While the rate of increase is fairly well marked in the lower numbers there are noticeable discrepancies among the higher magnitudes, and to some extent the table demonstrates the same anomaly as that found in Scheiner's catalogue. I have attempted to determine the average diameter that corresponds to Scheiner's assigned magnitudes, as it would be obviously desirable to continue Scheiner's scale of magnitudes. The results are as follows :

Scheiner's Magnitude.	Yerkes Diameter.	No. of Stars.
12·4	65·6	28
12·5	61·4	38
12·7	44·9	56
12·8	37·1	78
13·0	24·4	139
13·5	22·9	106
14	19·1	90

Scheiner has marked two stars as of 12·2 and 11·7 magnitudes respectively ; but the diameters of these fall below the average of 12·4, and two others are described as 13·8 magnitude. These also have been excluded from the list. If these magnitudes and diameters be plotted to scale the result points, as I have suggested, to the existence of two scales, or two distinct methods of measuring the diameter. It does not seem practical to draw a curve, even if a curve were legitimate, through the points. The diameters of the brighter stars lie on one straight line, and the diameters of the fainter on another. This is shown on the accompanying diagram where a linear relation apparently exists between both the brighter and the fainter stars. Four hours'



Comparison of Measured Diameter with conventional Magnitudes.

exposure with the Yerkes telescope would undoubtedly impress the images of fainter stars than those of the fourteenth magnitude, but the effect of the yellow screen is not known. But considering that Scheiner must have included some stars below the fourteenth magnitude in his faintest group, and after allowing for the over-estimation of the diameters of the faintest stars on the Yerkes plate, it seems not improbable that the number of stars brighter than the fifteenth magnitude does not greatly exceed 2,000.

The Distribution of Stars in the Cluster.

A question of greater interest seems to be to attempt to determine the law of density in the cluster itself, and the larger number of stars with which we have to deal than were at the command of Professor Scheiner makes the attempt a little more promising. There is, of course, some difficulty in deriving the number of stars in a definite area of various distances from the centre, but a tolerable approximation has been effected by dividing each square of the réseau (53″ of side) into four equal portions, and counting the number of stars in each quarter. These numbers were plotted in a diagram from which could be obtained the average number in a small square at uniform distances from the centre. At the centre of the cluster the number of stars in a square whose side was approximately twenty-seven seconds was sixty-five, and the cluster has been assumed to extend in the directions x and y to six minutes from the centre. At this distance about one star was found in each réseau square, and therefore the influence of the cluster may be supposed to have disappeared. In this way we have the apparent distribution of the stars when projected on a plane at right angles to the line of sight.

On the assumption of general radial symmetry let $f(r)dv$ be the number of stars in a small volume dv situated at a distance r from the centre of the cluster. Let x be the projection of r on the plane at right angles to the line of sight, and θ the angle between r and x. Then if a cylinder of small section $d\alpha$ be taken through the cluster with its axis parallel to the line of sight and at a distance x from the centre of the cluster, the number of stars to be seen within it is

$$\int f(r)d\alpha d(x \tan \theta)$$

where $x = r \cos \theta$ and the limits of the integral correspond to the boundary of the cluster. Eliminating θ and equating to $\phi(x)d\alpha$, the number of stars within the area $d\alpha$ in the apparent distribution, we get

$$\phi(x) = 2\int_x^1 rf(r)\frac{dr}{\sqrt{r^2-x^2}}$$

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the radius of the cluster being taken to be unity. If r be eliminated instead of θ, we get

$$\phi(x) = 2x\int_0^{\theta'} \sec^2 \theta f(x \sec \theta) d\theta$$

Here the upper limit θ' corresponds to the boundary and is a function of x when the radius of the cluster is finite. If the radius can be considered infinite the upper limit becomes $\frac{1}{2}\pi$. Hence if $f(r)$ can be expressed in the form $\Sigma a_n r^{-n}$, $\phi(x)$ has the corresponding form $\Sigma A_n x^{-n+1}$, where

$$A_n = 2a_n \int_0^{\frac{\pi}{2}} \cos^{n-2} \theta d\theta$$

When the law of density is expressed by one term only, this can easily be found ; for in this case

$$\log \phi(x) = \log A_n - (n-1) \log x$$

which gives a straight line when plotted with $\log x$ and $\log \phi(x)$ as coordinates. In the present case the graph shows distinct curvature, indicating that the law of density cannot be well represented by this simple form. The effect of a finite boundary to the cluster was next considered, but without an entirely satisfactory result. It seems probable that if the law of density is of this type, it requires more than one term for its expression.

When the radius of the cluster is supposed to be finite, a simple type of the law of density is represented by some positive power of the distance from the boundary. The radius being taken as unity, this law is expressed by $f(r) = (1 - r)^n$.

In this case

$$\phi(x) = 2\int_x^1 r(1-r)^n \frac{dr}{\sqrt{r^2-x^2}}$$

$$= 2\left[(1-r)^n \sqrt{r^2-x^2}\right]_x^1 + 2\int_x^1 n(1-r)^{n-1} \sqrt{r^2-x^2} dr$$

where the first term on the right vanishes at both limits. Let

$$Z_n = \int_n^1 (1-r)^n \frac{dr}{\sqrt{r^2-x^2}}$$

The equality of the above integrals shows that

$$Z_n - Z_{n+1} = n[Z_{n+1} - 2Z_n + (1-x^2)Z_{n-1}]$$

or

$$(n+1)Z_{n+1} - (2n+1)Z_n + n(1-x^2)Z_{n-1} = 0$$

Let

$$Y_n = Z_n - Z_{n+1}$$

so that

$$-(n+1)Y_n + nY_{n-1} - nx^2 Z_{n-1} = 0$$

$$-nY_{n-1} + (n-1)Y_{n-2} - (n-1)x^2 Z_{n-2} = 0$$

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Then the result of eliminating the Z functions becomes

$$(n^2-1)Y_n-n(2n-1)Y_{n-1}+n(n-1)(1-x^2)Y_{n-2}=0$$

Now $\phi(x) = 2Y_n$ and

$$Y_1=\tfrac{1}{2}a-\tfrac{1}{2}b,\ Y_2=\tfrac{1}{3}(2x^2+1)a-b$$

where

$$a=(1-x^2)^{\frac{1}{2}},\quad b=\tfrac{1}{2}x^2\log_e(1+a)/(1-a)$$

When $x=0$, $Y_1=\frac{1}{2}$ and $Y_2=\frac{1}{3}$, which shows that $Y_n=\dfrac{1}{n+1}$, for the solution of the difference equation in this case is clearly

$$Y_n=A/(n+1)+Bn$$

Hence if we put $\phi_n(x)=(n+1)Y_n$ we make the central density in the apparent distribution unity. The difference equation becomes

$$(n-1)\phi_n-(2n-1)\phi_{n-1}+n(1-x^2)\phi_{n-2}=0$$

with the initial values

$$\phi_0=a,\ \phi_1=a-b,\ \phi_2=(2x^2+1)a-3b$$

These can be calculated numerically for convenient intervals of x, and the values of ϕ_3, ϕ_4 . . . can be easily deduced in succession by means of the difference equation. When the process has to be carried to considerable values of n, it is of course necessary to begin with a greater number of decimal places than are ultimately required on account of the accumulation of numerical error. In this way the following table has been calculated :

n/x.	0·0.	·1.	·2.	·3.	·4.	·5.	·6.	·7.	·8.	·9.	1·0.
0	1·000	0·995	·980	·954	·917	·866	·800	·714	·600	·436	·000
1	1·000	·965	·888	·785	·666	·537	·405	·275	·156	·057	·000
2	1·000	·925	·783	·620	·458	·311	·190	·097	·037	·007	·000
3	1·000	·880	·679	·477	·305	·174	·085	·033	·008	·001	·000
4	1·000	·831	·582	·362	·200	·095	·038	·011	·002		
5	1·000	·782	·494	·271	·129	·051	·016	·003			
6	1·000	·733	·417	·202	·083	·027	·007				
7	1·000	·684	·350	·149	·052	·014	·003				
8	1·000	·637	·292	·109	·033	·007	·001				
9	1·000	·592	·243	·080	·021	·002	·001				
10	1·000	·549	·202	·058	·013						
11	1·000	·509	·167	·042	·007						
12	1·000	·470	·137	·030	·004						·000

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When these values are plotted they give curves which represent the general character of the observed distribution of density. For a good agreement a large value of n seems necessary. As n increases the curves tend to similarity, and the uncertainty in the radius of the cluster makes it impossible to attach any great weight to the determination of n. With the diameter already mentioned a fair representation is obtained by assuming $n = 11$. Unfortunately it is not easy to show the comparison graphically except on a very large scale, as the variation from straight line is small in the greater part of the curve, and the various curves have a tendency to coalesce. But assuming a central density of sixty-five stars in a unit of space, the number of stars read from the curves can be compared numerically with the values computed from the above table.

Distance from Centre.	No. of Stars Observed.	No. of Stars $n=8$.	No. of Stars $n=9$.	No. of Stars $n=10$.	No. of Stars $n=11$.
″ 0	65·0	65·0	65·0	65·0	65·0
45	33·4	41·4	38·5	35·7	33·1
93	12·7	19·0	15·8	13·1	10·9
136	5·0	7·1	5·2	3·8	2·7
188	2·3	2·2	1·4	0·8	0·5
258	1·1	0·5	0·1		

The agreement is not very satisfactory in any case, but it seems safe to say that while n may be as high as 11, it is not less than 8. Professor Geo. E. Hale has offered me a plate on which the number of images recorded is much larger than on the plate here discussed, and I hope to return to this question again.

The discussion as far as it goes affords an example of the spirit of devolution of which we have heard something of late, and I acknowledge my indebtedness to the Directors of the Yerkes and Oxford Observatories, and to others, who have enabled me to carry out this investigation.

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The Solar Rotation Period from Greenwich Sun-spot Measures, 1879–1901. By E. Walter Maunder and A. S. D. Maunder.

1. *Material Employed.*

The material employed in the following paper is much more ample than has been used in any previous discussion of the solar rotation. It is drawn from the Photoheliographic Results published in the annual volumes of the *Greenwich Observations*, and especially from the ledgers of sun-spots. From the 4700 spot-groups catalogued in these results for the twenty-three years (1879–1901) we have taken every group which persisted